Lecture 13: Eilenberg-MacLane Space

$\pi_n(S^n)$ and Degree

Theorem (Freudenthal Suspension Theorem)

The suspension map

$$\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$$

is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$.

Freudenthal Suspension Theorem holds similarly replacing S^n by general $(n-1)$-connected space.

Proposition

$\pi_n(S^n) \simeq \mathbb{Z}$ for $n \geq 1$.

Proof.

Freudenthal Suspension Theorem reduces to show $\pi_2(S^2) \simeq \mathbb{Z}$. This follows from the Hopf fibration

$$S^1 \to S^3 \to S^2.$$

□

Definition

Given $f: S^n \to S^n$, its class $[f] \in \mathbb{Z}$ under the above isomorphism is called the degree of f.

Eilenberg-MacLane Space

Definition

An Eilenberg-MacLane Space of type (G, n) is a CW complex X such that

$$\pi_n(X) \simeq G \quad \text{and} \quad \pi_k(X) = 0 \quad \text{for} \quad k \neq n.$$

Here G is abelian if $n > 1$.

As we will show next, Eilenberg-MacLane Space of any type (G, n) exists and is unique up to homotopy. It will be denoted by $K(G, n)$.

The importance of $K(G, n)$ is that it is the representing space for cohomology functor with coefficients in G

$$H^n(X; G) \simeq [X, K(G, n)] \quad \text{for any CW complex } X.$$

Theorem

Eilenberg-MacLane Spaces exist.

Proof: We prove the case for $n \geq 2$. There exists an exact sequence

$$0 \to F_1 \to F_2 \to G \to 0$$

where F_1, F_2 are free abelian groups. Let B_i be a basis of F_i. Let

$$A = \bigvee_{i \in B_1} S^n, \quad B = \bigvee_{j \in B_2} S^n.$$

A, B are $(n-1)$-connected and $\pi_n(A) = F_1, \pi_n(B) = F_2$.

Using the degree map, we can construct

$$f: A \to B$$

such that $\pi_n(A) \to \pi_n(B)$ realizes the map $F_1 \to F_2$. Let X be obtained from B by attaching $(n+1)$-cells via f. Then X is $(n-1)$-connected and $\pi_n(X) = G$.

Now we proceed as in the proof of CW Approximation Theorem to attach cells of dimension $\geq (n+2)$ to kill all higher homotopy groups of X to get $K(G, n)$. □

Theorem

Let X be an $(n-1)$-connected CW complex. Let Y be an Eilenberg-MacLane Space of type (G, n). Then the map

$$\phi : [X, Y] \to \mathrm{Hom}(\pi_n(X), \pi_n(Y)), \quad f \to f_*$$

is a bijection. In particular, any two Eilenberg-MacLane Spaces of type (G, n) are homotopy equivalent.

Proof

Let us first do two simplifications.

First, we can find a CW complex Z and a weak homotopy equivalence $g : Z \to X$ such that the n-skeleton of Z is

$$Z^n = \bigvee_{j \in J} S^n.$$

By Whitehead Theorem, g is also a homotopy equivalence. So we can assume the n-skeleton of X is

$$X^n = \bigvee_{j \in J} S^n.$$

Secondly, let X^{n+1} be the $(n+1)$-skeleton of X. Then

$$\pi_n(X) = \pi_n(X^{n+1}).$$

Let $f\colon X \to Y$. Since X is obtained from X^{n+1} by attaching cells of dimension $\geq n+2$ and $\pi_k(Y) = 0$ for all $k > n$, any map $X^{n+1} \to Y$ can be extended to $X \to Y$. So the natural map

$$[X, Y] \to [X^{n+1}, Y]$$

is a surjection.

Now assume $f\colon X \to Y$ such that its restriction to X^{n+1} is null-homotopic. Since $X^{n+1} \subset X$ is a cofibration, f is homotopic to a map which shrinks the whole X^{n+1} to a point. Since $\pi_k(Y) = 0$ for all $k > n$, f is further null-homotopic. This implies that

$$[X, Y] \to [X^{n+1}, Y]$$

is a bijection.

So we can also assume $X = X^{n+1}$ has dimension at most $n + 1$.

Assume X is obtained from X^n by attaching $(n+1)$-cells via

$$\chi : \bigvee_{i \in I} S^n \to \bigvee_{j \in J} S^n.$$

We now proceed to show

$$\phi : [X, Y] \to \mathrm{Hom}(\pi_n(X), \pi_n(Y)), \quad f \to f_*$$

is a bijection.

Injectivity of ϕ. Assume $f: X \to Y$ such that $\phi(f) = 0$. Then the restriction of f to

$$X^n = \bigvee_{j \in J} S^n \to Y$$

is null-homotopic. Since $X^n \hookrightarrow X$ is a cofibration, f is homotopic to a map which shrinks X^n to a point, so can be viewed as a map

$$\bigvee_{i \in I} S^{n+1} \to Y.$$

Since $\pi_{n+1}(Y) = 0$, this map is also null-homotopic. So $[f] = 0$.

Surjectivity of ϕ. Let $g : \pi_n(X) \to \pi_n(Y)$ be a group homomorphism. Since

$$j : \pi_n(X^n) \to \pi_n(X)$$

is surjective and $\pi_n(X^n)$ is free, we can find a map

$$f_n : X^n \to Y$$

such that $f_{n*} : \pi_n(X^n) \to \pi_n(Y)$ coincides with $g \circ j$. By construction, $f_n \circ \chi$ is null-homotopic, so we can extend f_n to a map $f : X \to Y$ which gives the required group homomorphism.

Now assume we have two Eilenberg-MacLane Spaces Y_1, Y_2 of type (G, n). We have the identification

$$[Y_1, Y_2] = \mathrm{Hom}(\pi_n(Y_1), \pi_n(Y_2)).$$

Then a group isomorphism $\pi_n(Y_1) \to \pi_n(Y_2)$ gives a homotopy equivalence $Y_1 \to Y_2$. $\square$

Remark

A classical result of Milnor says the loop space of a CW complex is homotopy equivalent to a CW complex. Since for any X, we have $\pi_k(\Omega X) = \pi_{k+1}(X)$. Therefore

$$\Omega K(G, n) \simeq K(G, n-1).$$

Example

$S^1 = K(\mathbb{Z}, 1)$ and $\bigvee_{i=1}^{m} S^1 = K(Z^m, 1)$.

Example

We have natural embeddings

$$\mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \cdots \mathbb{CP}^{n-1} \subset \mathbb{CP}^n \subset \cdots \subset \mathbb{CP}^\infty$$

and

$$S^1 \subset S^3 \subset \cdots S^{2n-1} \subset S^{2n+1} \subset \cdots \subset S^\infty.$$

This gives rise to the fibration

$$S^1 \to S^\infty \to \mathbb{CP}^\infty.$$

This shows

$$\mathbb{CP}^\infty = K(\mathbb{Z}, 2).$$

Example

A knot is an embedding $K : S^1 \hookrightarrow S^3$. Let $G = \pi_1(S^3 - K)$. Then

$$S^3 - K = K(G, 1).$$

Postnikov Tower

Postnikov tower for a space is a decomposition dual to a cell decomposition. In the Postnikov tower description of a space, the building blocks of the space are Eilenberg-MacLane spaces.

Definition

A Postnikov tower of a path-connected space X is a tower diagram

$$\cdots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \ .$$

with a sequence of compatible maps $f_n : X \to X_n$ satisfying

1. f_n induces an isomorphism $\pi_k(X) \to \pi_k(X_n)$ for any $k \leq n$
2. $\pi_k(X_n) = 0$ for $k > n$
3. each $X_n \to X_{n-1}$ is a fibration with fiber $K(\pi_n(X), n)$.

$$\begin{array}{ccccc} & & X_n & \longleftarrow & K(\pi_n(X), n) \\ & {\scriptstyle f_n}\nearrow & \downarrow & & \\ X & \xrightarrow[f_{n-1}]{} & X_{n-1} & & \end{array}$$

X_n is called a n-th Postnikov approximation of X.

Note that if X is $(n-1)$-connected, then $X_n = K(\pi_n(X), n)$. In general, a Postnikov tower can be viewed as an approximation of a space by twisted product of Eilenberg-MacLane spaces.

Theorem

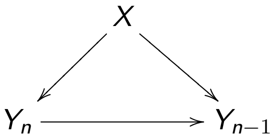
Postnikov Tower exists for any connected CW complex.

Proof

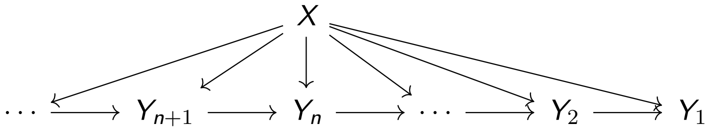
Let X be a connected CW complex. Let us construct Y_n which is obtained from X by successively attaching cells of dimensions $n+2, n+3, \cdots$ to kill homotopy groups $\pi_k(X)$ for $k > n$. Then we have a CW subcomplex $X \subset Y_n$ such that

$$\begin{cases} \pi_k(X) \to \pi_k(Y_n) \text{ is an isomorphism} & \text{if} \quad k \leq n \\ \pi_k(Y_n) = 0 & \text{if} \quad k > n. \end{cases}$$

Since $\pi_k(Y_{n-1}) = 0$ for $k \geq n$, we can extend the map $X \to Y_{n-1}$ to a map $Y_n \to Y_{n-1}$ making the following diagram commutative



In this way we find a tower diagram



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Now we can replace $Y_2 \to Y_1$ by a fibration, and then similarly adjust $Y_3, Y_4, \cdots$ successively to end up with

$$\begin{array}{ccccccccccc}
 & & & & & X & & & & & \\
 & \swarrow & & \swarrow & & \downarrow & & \searrow & & \searrow & \searrow \\
\cdots & \longrightarrow & Y_{n+1} & \longrightarrow & Y_n & \longrightarrow & \cdots & \longrightarrow & Y_2 & \longrightarrow & Y_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & X_{n+1} & \longrightarrow & X_n & \longrightarrow & \cdots & \longrightarrow & X_2 & \longrightarrow & X_1 = Y_1
\end{array} .$$

such that each $X_n \to X_{n-1}$ is a fibration with fiber F_n.

Since X_n is homotopy equivalent to Y_n, we have

$$\begin{cases} \pi_k(X_n) = \pi_k(X) & \text{if} \quad k \le n \\ \pi_k(X_n) = 0 & \text{if} \quad k > n. \end{cases}$$

Then the long exact sequence of homotopy groups associated to the fibration $F_n \to X_n \to X_{n-1}$ implies

$$F_n \simeq K(\pi_n(X), n).$$

□

Whitehead Tower

Whitehead Tower is a sequence of fibrations that generalize the universal covering of a space.

Theorem (Whitehead Tower)

Let X be a connected CW complex. There is a sequence of maps

$$\cdots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X$$

where each map $X_n \to X_{n-1}$ is a fibration with fiber $K(\pi_n(X), n-1)$. Each X_n satisfies

$$\begin{cases} \pi_k(X_n) \to \pi_k(X) \text{ is an isomorphism} & \text{if} \quad k > n \\ \pi_k(X_n) = 0 & \text{if} \quad k \leq n. \end{cases}$$

Proof

Let $Y_1 \simeq K(\pi_1(X), 1)$ be obtained from X by successively attaching cells to kill $\pi_k(X)$ for $k > 1$. Let $j_1 : X \subset Y_1$ and $X_1 = F_{j_1}$ be the homotopy fiber. Then we have a fibration

$$\begin{array}{ccc} \Omega Y_1 & \longrightarrow & X_1 \\ & & \downarrow \\ & & X \end{array}$$

Note that $\Omega Y_1 \simeq K(\pi_1(X), 0)$ and $\pi_1(X_1) = 0$. So X_1 can be viewed as the universal cover of X up to homotopy equivalence.

Similarly, assume we have constructed the Whitehead Tower up to X_n. Let $Y_n \simeq K(\pi_n(X), n)$ be obtained from X_n by killing homotopy groups $\pi_k(X)$ for $k > n$. Let $j_n : X_n \subset Y_n$. Then we define

$$X_{n+1} = F_{j_n}$$

to be the homotopy fiber.

Repeating this process, we obtain the Whitehead Tower. □